Theoretical Foundations I: Structure of Rough Approximations

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Relations

- Let U be a nonempty set, called often **universe** (or **universe of discourse**). This is the set of elements (or objects) we are interested in.
- Let R be a **binary relation** on U representing some knowledge about the elements in U. Binary relation R consists of ordered pairs (a, b) such that $a \, R \, b$.
- The relation R is interpreted to represent some knowledge about the objects in U.

Example

Let us consider a datatable (database table, Excel table, etc.) representing some information about human beings. The table may contain columns such that **weight**, **height**, **age**, **gender**, **home town**, etc. Two objects are R-related if values for all the above attributes are the same.

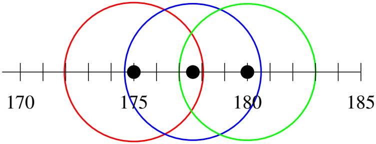
Relations

Example

In the previous example, we defined the relation R in such a way that two objects are R-related, if they have exactly the same values for all attributes.
We may also define a relation such that two objects are R-related if their values are "close enough". For instance,

$$a\, R\, b \iff |\text{height}(a) - \text{height}(b)| \leq \varepsilon,$$

where ε is a suitable **threshold**. In the next figure, ε equals 3 cm.



Relations

Example

The statement "a is preferred to b" is generally understood to mean that someone chooses a over b. In this case, the universe U consists of different choices and $a\,R\,b$ tells that a is preferred over b. For instance, someone could say that: "I prefer sleeping over running".

Rough approximations based on binary relations

Let R be a binary relation on U, and let us denote for all $x \in U$,

$$R(x) = \{y \in U \mid x\,R\,y\}$$

The **upper approximation** of $X \subseteq U$ is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\}$$

and the **lower approximation** of X is

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}$$

The set $B(X) = X^{\blacktriangle} \setminus X^{\blacktriangledown}$ is the **boundary** of X

Rough approximations

The **lower approximation of** $X^\blacktriangledown$ can be viewed as the set of elements that **certainly are in** X when **observed through the knowledge** R, because all elements R-related to them are in X.

The **upper approximation** $X^\blacktriangle$ can be viewed as the set of elements that **possible are in** X, because in X is at least one element R-related to them.

Proposition

If R is a binary relation on U, then following assertions hold.

(a) *The maps* $^{\blacktriangledown}$ *and* $^{\blacktriangle}$ *are mutually dual, i.e.* $X^{\blacktriangledown c} = X^{c\blacktriangle}$ *and* $X^{\blacktriangle c} = X^{c\blacktriangledown}$

(b) *The boundary of any set is equal to the boundary of its complement.*

(c) *The maps* $^{\blacktriangledown}$ *and* $^{\blacktriangle}$ *are order-preserving.*

Proof.

(a) $x \in X^{\blacktriangledown c} \Leftrightarrow x \notin X^{\blacktriangledown} \Leftrightarrow R(x) \not\subseteq X \Leftrightarrow R(x) \cap X^c \neq \emptyset \Leftrightarrow x \in X^{c\blacktriangle}$. Further, $X^{\blacktriangle c} = X^{cc\blacktriangle c} = X^{c\blacktriangledown cc} = X^{c\blacktriangledown}$.

(b) $B(X) = X^{\blacktriangle} \setminus X^{\blacktriangledown} = X^{\blacktriangle} \cap X^{\blacktriangledown c} = X^{c\blacktriangledown c} \cap X^{c\blacktriangle} = X^{c\blacktriangle} \setminus X^{c\blacktriangledown} = B(X^c)$.

(c) Suppose $X \subseteq Y$. If $x \in X^{\blacktriangledown}$, then $R(x) \subseteq X \subseteq Y$, i.e. $x \in Y^{\blacktriangledown}$. If $x \in X^{\blacktriangle}$, then $R(x) \cap Y \supseteq R(x) \cap X \neq \emptyset$, i.e. $x \in Y^{\blacktriangle}$. $\square$

We denote

$$\wp(U)^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\} \quad \text{and} \quad \wp(U)^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}.$$

Proposition

The ordered sets $(\wp(U)^{\blacktriangle}, \subseteq)$ *and* $(\wp(U)^{\blacktriangledown}, \subseteq)$ *are dually isomorphic.*

Proof.

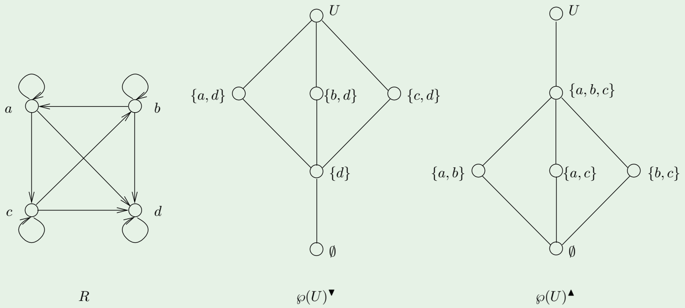
We show that the map $\phi\colon X^{\blacktriangle} \mapsto X^{c\blacktriangledown}$ is an order-isomorphism between $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \supseteq)$.

$X^{\blacktriangle} \subseteq Y^{\blacktriangle} \Leftrightarrow \phi(X^{\blacktriangle}) = X^{c\blacktriangledown} = X^{\blacktriangle c} \supseteq Y^{\blacktriangle c} = Y^{c\blacktriangledown} = \phi(Y^{\blacktriangle})$.
Thus, ϕ is an order-embedding.

If $X^{\blacktriangledown} \in \wp(U)^{\blacktriangledown}$, then $\phi(X^{c\blacktriangle}) = X^{cc\blacktriangledown} = X^{\blacktriangledown}$, i.e., ϕ is onto. □

Example

Let $U = \{a, b, c, d\}$.



- $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ seem to be lattices (we will study in detail what kind of lattices these are).
- $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not distributive, because they contain $\mathbf{M_3}$ as a sublattice.
- These lattices are not complemented.

- We denote by R^{-1} the inverse relation of R and

$$R^{-1}(x) = \{y \mid y\, R\, x\}.$$

- We define

$$X^{\triangle} = \{x \in U \mid R^{-1}(x) \cap X \neq \emptyset\}$$

and

$$X^{\triangledown} = \{x \in U \mid R^{-1}(x) \subseteq X\}.$$

- Note that

$$\{x\}^{\blacktriangle} = \{y \mid R(y) \cap \{x\} \neq \emptyset\} = \{y \mid x \in R(y)\} = R^{-1}(x)$$

- Similarly, $\{x\}^{\triangle} = R(x)$

Galois connection

- **Galois connections** are pairs of maps which enable us to move back and forth between two ordered sets.
- Galois connections tie different structures firmly and when a Galois connection is found between two structures, we immediately know that they have much in common.
- After an element is mapped to the other structure and back, a certain stability is reached in such a way that further mappings give the same results.
- We will show that the pairs $({}^{\blacktriangle}, {}^{\triangledown})$ and $({}^{\vartriangle}, {}^{\blacktriangledown})$ form Galois connections. Several observations and properties of rough approximations follow from this.

Galois connections

Definition ("flip-flop" property)

For two partially ordered sets $(P, \leq)$ and $(Q, \leq)$, a pair (f, g) of maps $f \colon P \to Q$ and $g \colon Q \to P$ is called a **Galois connection** between P and Q if for all $p \in P$ and $q \in Q$,

$$f(p) \leq q \iff p \leq g(q).$$

Such a mapping f is sometimes called **residuated mapping**. The mapping g is called the **residual mapping** of f.

Lemma

The pair (f, g) is a Galois connection between P and Q iff

(a) $p \leq (g \circ f)(p)$ *for all* $p \in P$ *and* $(f \circ g)(q) \leq q$ *for all* $q \in Q$

(b) *the maps f and g are order-preserving*

Galois connections I

Let (f, g) be a Galois connection between two complete lattices P and Q.

1. $f \circ g \circ f = f$ and $g \circ f \circ g = g$.
2. The map $g \circ f$ is a (lattice-theoretical) **closure operator** on P (extensive, order-preserving, idempotent) and the set of $g \circ f$-closed elements is $g(Q)$, that is, $(g \circ f)(P) = g(Q)$
3. The map $f \circ g$ is a (lattice-theoretical) **interior operator** on Q (inflationary, order-preserving, idempotent) and the set of $f \circ g$-open elements is $f(P)$, that is, $(f \circ g)(Q) = f(P)$.
4. The map f is a complete join-morphism and g is a complete meet-morphism, that is,

$$f\left(\bigvee S\right) = \bigvee f(S) \quad \text{and} \quad g\left(\bigwedge T\right) = \bigwedge g(T)$$

for $S \subseteq P$ and $T \subseteq Q$.

Galois connections II

5. The image sets $f(P)$ and $g(Q)$ are order-isomorphic.
6. The ordered set $f(P)$ is a complete lattice such that for all $S \subseteq f(P)$ $(\subseteq Q)$,

$$\bigvee S = \bigvee_Q S \quad \text{and} \quad \bigwedge S = f\big(g\big(\bigwedge_Q S\big)\big) = f\big(\bigwedge_P g(S)\big).$$

7. The ordered set $g(Q)$ is a complete lattice such that for all $S \subseteq g(Q)$ $(\subseteq P)$,

$$\bigvee S = g\big(f\big(\bigvee_P S\big)\big) = g\big(\bigvee_Q f(S)\big) \quad \text{and} \quad \bigwedge S = \bigwedge_P S.$$

Galois connections of rough approximations

The ordered set $(\wp(U), \subseteq)$ is a complete lattice such that

$$\bigvee \mathcal{H} = \bigcup \mathcal{H} \quad \text{and} \quad \bigwedge \mathcal{H} = \bigcap \mathcal{H}$$

for all $\mathcal{H} \subseteq \wp(U)$.

Proposition

For any binary relation R on U, the pairs $(^\blacktriangle, ^\triangledown)$ and $(^\vartriangle, ^\blacktriangledown)$ are order-preserving Galois connections on $(\wp(U), \subseteq)$.

Proof.

As noted, the maps $X \mapsto X^\blacktriangle$ and $X \mapsto X^\triangledown$ are order-preserving.

If $x \in X^{\triangledown\blacktriangle}$, there exists $y \in X^\triangledown$ such that $(x, y) \in R$. Because $y \in X^\triangledown$ and $(y, x) \in R^{-1}$, we have $x \in X$. Hence, $X^{\triangledown\blacktriangle} \subseteq X$. This also gives $X^{\vartriangle\blacktriangledown c} = X^{c\triangledown\blacktriangle} \subseteq X^c$, i.e., $X \subseteq X^{\vartriangle\blacktriangledown}$. □

What this then means? I

1. $X^{\blacktriangle\triangledown\blacktriangle} = X^{\blacktriangle}$, $X^{\vartriangle\blacktriangledown\vartriangle} = X^{\vartriangle}$, $X^{\triangledown\blacktriangle\triangledown} = X^{\triangledown}$, $X^{\blacktriangledown\vartriangle\blacktriangledown} = X^{\blacktriangledown}$

2 a. The map $X \mapsto X^{\blacktriangle\triangledown}$ is a closure operator. The set of closed sets is $\wp(U)^{\triangledown}$, i.e. $\{X^{\blacktriangle\triangledown} \mid X \subseteq U\} = \wp(U)^{\triangledown}$.

2 b. The map $X \mapsto X^{\vartriangle\blacktriangledown}$ is a closure operator. The set of closed sets is $\wp(U)^{\blacktriangledown}$, i.e. $\{X^{\vartriangle\blacktriangledown} \mid X \subseteq U\} = \wp(U)^{\blacktriangledown}$.

3 a. The map $X \mapsto X^{\triangledown\blacktriangle}$ is an interior operator. The set of open sets is $\wp(U)^{\blacktriangle}$, i.e. $\{X^{\triangledown\blacktriangle} \mid X \subseteq U\} = \wp(U)^{\blacktriangle}$.

3 b. The map $X \mapsto X^{\blacktriangledown\vartriangle}$ is an interior operator. The set of open sets is $\wp(U)^{\vartriangle}$, i.e. $\{X^{\blacktriangledown\vartriangle} \mid X \subseteq U\} = \wp(U)^{\vartriangle}$.

What this then means? II

4 a. For $\mathcal{H} \subseteq \wp(U)$:
$$\Big(\bigcup_{X\in\mathcal{H}} X\Big)^{\blacktriangle} = \bigcup_{X\in\mathcal{H}} X^{\blacktriangle} \text{ and } \Big(\bigcup_{X\in\mathcal{H}} X\Big)^{\vartriangle} = \bigcup_{X\in\mathcal{H}} X^{\vartriangle}$$
Note that this implies that $X^{\blacktriangle} = \bigcup_{x\in X}\{x\}^{\blacktriangle} = \bigcup_{x\in X} R^{-1}(x)$ and $X^{\vartriangle} = \bigcup_{x\in X}\{x\}^{\vartriangle} = \bigcup_{x\in X} R(x)$

4 b. $$\Big(\bigcap_{X\in\mathcal{H}} X\Big)^{\blacktriangledown} = \bigcap_{X\in\mathcal{H}} X^{\blacktriangledown} \text{ and } \Big(\bigcap_{X\in\mathcal{H}} X\Big)^{\triangledown} = \bigcap_{X\in\mathcal{H}} X^{\triangledown}$$

5. $\wp(U)^{\blacktriangle} \cong \wp(U)^{\triangledown}$ and $\wp(U)^{\vartriangle} \cong \wp(U)^{\blacktriangledown}$

What this then means? III

6 a. The ordered set $(\wp(U)^{\blacktriangle}, \subseteq)$ is a complete lattice such that

$$\bigvee_{X \in \mathcal{H}} X^{\blacktriangle} = \bigcup_{X \in \mathcal{H}} X^{\blacktriangle} \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^{\blacktriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle} \Big)^{\triangledown \blacktriangle}$$

6 b. The ordered set $(\wp(U)^{\vartriangle}, \subseteq)$ is a complete lattice such that

$$\bigvee_{X \in \mathcal{H}} X^{\vartriangle} = \bigcup_{X \in \mathcal{H}} X^{\vartriangle} \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^{\vartriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\vartriangle} \Big)^{\blacktriangledown \vartriangle}$$

What this then means? IV

7 a. The ordered set $(\wp(U)^{\blacktriangledown}, \subseteq)$ is a complete lattice such that

$$\bigwedge_{X \in \mathcal{H}} X^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown} \quad \text{and} \quad \bigvee_{X \in \mathcal{H}} X^{\blacktriangledown} = \Big(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown} \Big)^{\vartriangle \blacktriangledown}$$

7 b. The ordered set $(\wp(U)^{\triangledown}, \subseteq)$ is a complete lattice such that

$$\bigwedge_{X \in \mathcal{H}} X^{\triangledown} = \bigcap_{X \in \mathcal{H}} X^{\triangledown} \quad \text{and} \quad \bigvee_{X \in \mathcal{H}} X^{\triangledown} = \Big(\bigcup_{X \in \mathcal{H}} X^{\triangledown} \Big)^{\blacktriangle \triangledown}$$

Different types of relations

A binary relation R on U is said to be:

- **left-total** if for all $x \in U$, the exists $y \in U$ such that $x \, R \, y$.
- **reflexive** if for all $x \in U$, $x \, R \, x$.
- **symmetric** if $x \, R \, y$ implies $y \, R \, x$.
- **antisymmetric** if $x \, R \, y$ and $y \, R \, x$ imply $x = y$.
- **transitive** if $x \, R \, y$ and $y \, R \, z$ imply $x \, R \, z$.
- **a tolerance** if it is reflexive and symmetric
- **a quasiorder (or a preorder)** if it is reflexive and transitive
- **a partial order** if it is reflexive, antisymmetric and transitive
- **an equivalence** if it is reflexive, symmetric and transitive

Example

Let E be an equivalence on U such that $\{a, b\}$ and $\{c, d\}$ are E-equivalence classes.
We know that $X^{\blacktriangle} \cup Y^{\blacktriangle} = (X \cup Y)^{\blacktriangle}$, but $X^{\blacktriangle} \cap Y^{\blacktriangle} \supseteq (X \cap Y)^{\blacktriangle}$, and the inclusion can be proper!
Let $X = \{a, c\}$ and $Y = \{b, d\}$. Then $X^{\blacktriangle} = U$ and $Y^{\blacktriangle} = U$, and $X^{\blacktriangle} \cap Y^{\blacktriangle} = U$, but $(X \cap Y)^{\blacktriangle} = \emptyset^{\blacktriangle} = \emptyset$.
Analogously, we have $X^{\blacktriangledown} \cap Y^{\blacktriangledown} = (X \cap Y)^{\blacktriangledown}$, but $X^{\blacktriangledown} \cup Y^{\blacktriangledown} \subseteq (X \cup Y)^{\blacktriangledown}$.
Also this inclusion can be proper, because $X^{\blacktriangledown} = \emptyset$, $Y^{\blacktriangledown} = \emptyset$, and $X^{\blacktriangledown} \cup Y^{\blacktriangledown} = \emptyset$. But: $(X \cup Y)^{\blacktriangledown} = U^{\blacktriangledown} = U$.

Correspondences: **left-total relations**

Proposition

If R is a binary relation on U, then the following are equivalent:

(a) *R is left-total;*

(b) *$X^{\blacktriangledown} \subseteq X^{\blacktriangle}$ for all $X \subseteq U$.*

Proof.

(a) $\Rightarrow$ (b): Let $x \in X^{\blacktriangledown}$. Then $R(x) \subseteq X$, which gives $R(x) \cap X = R(x) \neq \emptyset$, i.e., $x \in X^{\blacktriangle}$.

(b) $\Rightarrow$ (a): Assume that R is not left-total, i.e. $R(x) = \emptyset$ for some $x \in U$. This means that $x \in X^{\blacktriangledown}$ and $x \notin X^{\blacktriangle}$ for this particular x and for any set $X \subseteq U$, a contradiction! $\square$

Correspondences: **reflexive relations**

Proposition

TFAE:

(a) R *is reflexive;*

(b) $X \subseteq X^{\blacktriangle}$ *for all* $X \subseteq U$*;*

(c) $X^{\blacktriangledown} \subseteq X$ *for all* $X \subseteq U$*.*

Proof.

(a) $\Rightarrow$ (b): If $x \in X$, then $x \in R(x) \cap X \neq \emptyset$, i.e. $x \in X^{\blacktriangle}$.

(b) $\Rightarrow$ (c): $X^c \subseteq X^{c\blacktriangle} = X^{\blacktriangledown c}$ gives $X^{\blacktriangledown} \subseteq X$.

(c) $\Rightarrow$ (a): If R is not reflexive, there is $x \in U$ such that $(x, x) \notin R$. Let us consider the set $X = U \setminus \{x\}$. Now $(x, y) \in R$ implies $y \in X$. Thus, $x \in X^{\blacktriangledown}$ and $x \notin X$, a contradiction! □

Correspondences: **symmetric relations**

Proposition

TFAE:

(a) *R is symmetric;*

(b) *$(^\blacktriangle, ^\blacktriangledown)$ is a Galois connection on $(\wp(U), \subseteq)$.*

Proof.

(a) $\Rightarrow$ (b): If R is symmetric, then $X^\blacktriangle = X^\vartriangle$ and $X^\blacktriangledown = X^\triangledown$ for all $X \subseteq U$. Recall that $(^\blacktriangle, ^\triangledown)$ is a Galois connection.
(b) $\Rightarrow$ (a): If R is not symmetric, then for some $x, y \in U$, $(x, y) \in R$, but $(y, x) \notin R$. Let $X = \{x\}$. For all $z \in U$, $(y, z) \in R$ implies $z \notin X$. This gives $y \notin X^\blacktriangle$. Hence, $x \in X$ and $x \notin X^{\blacktriangle\blacktriangledown}$, a contradiction! □

Correspondences: **transitive relations**

Proposition

TFAE:

(a) *R is transitive;*

(b) *$X^{\blacktriangle\blacktriangle} \subseteq X^{\blacktriangle}$ for all $X \subseteq U$;*

(c) *$X^{\blacktriangledown} \subseteq X^{\blacktriangledown\blacktriangledown}$ for all $X \subseteq U$.*

Proof.

(a) $\Rightarrow$ (b): Let $x \in X^{\blacktriangle\blacktriangle}$. There is $y \in X^{\blacktriangle}$ such that $(x, y) \in R$. Since $y \in X^{\blacktriangle}$, there is $z \in X$ such that $(y, z) \in R$. So, also $(x, z) \in R$ and $x \in X^{\blacktriangle}$.

(b) $\Rightarrow$ (c): $X^{\blacktriangledown\blacktriangledown c} = X^{c\blacktriangle\blacktriangle} \subseteq X^{c\blacktriangle} = X^{\blacktriangledown c}$, which gives $X^{\blacktriangledown} \subseteq X^{\blacktriangledown\blacktriangledown}$.

(c) $\Rightarrow$ (a): If R is not transitive, there are $x, y, z \in U$ such that $(x, y) \in R$ and $(y, z) \in R$, but $(x, z) \notin R$. Let $X = U \setminus \{z\}$. Then for all $w \in U$, (x, w) implies $w \in X$. Thus, $x \in X^{\blacktriangledown}$. Obviously, $y \notin X^{\blacktriangledown}$ and hence $x \notin X^{\blacktriangledown\blacktriangledown}$, a contradiction! □

Correspondences for $^\vartriangle$ and $^\triangledown$

- Note that R is reflexive if and only if R^{-1} is reflexive,
- Similar conditions hold also for symmetry and transitivity.
- We can state similar correspondences between R and the operators $X \mapsto X^\vartriangle$ and $X \mapsto X^\triangledown$.
- However, with left-/right-total relations we have to make the following exception:

$$\begin{aligned}(\forall X \subseteq U)\, X^\triangledown \subseteq X^\vartriangle &\iff R^{-1} \text{ is left-total} \\ &\iff R \text{ is right-total}\end{aligned}$$

Properties of rough approximations: **tolerances**

Let R be a tolerance on U and $X, Y \subseteq U$.

(a) $X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$

(b) $({}^{\blacktriangle}, {}^{\blacktriangledown})$ is an order-preserving Galois connection on $(\wp(U), \subseteq)$:

$$X^{\blacktriangle} \subseteq Y \iff X \subseteq Y^{\blacktriangledown}$$

(c) $X^{\blacktriangle\blacktriangledown\blacktriangle} = X^{\blacktriangle}$ and $X^{\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangledown}$.

Proposition

Let (F, G) be a Galois connection on $(\wp(U), \subseteq)$. There exists a tolerance R on U such that F equals ${}^{\blacktriangle}$ and G equals ${}^{\blacktriangledown}$ if and only if the following conditions hold for all $x, y \in U$:

(i) $x \in F(\{x\})$;

(ii) $x \in F(\{y\})$ *implies* $y \in F(\{x\})$.

Lattice structures of approximations: **tolerances**

Let R be a tolerance.

(a) $(\wp(U)^{\blacktriangledown}, \subseteq)$ forms a complete lattice such that for $\mathcal{H} \subseteq \wp(U)$:

$$\bigvee_{X \in \mathcal{H}} X^{\blacktriangledown} = \Big(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown} \Big)^{\blacktriangle\blacktriangledown} \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}$$

(b) $(\wp(U)^{\blacktriangle}, \subseteq)$ forms a complete lattice such that for $\mathcal{H} \subseteq \wp(U)$:

$$\bigvee_{X \in \mathcal{H}} X^{\blacktriangle} = \bigcup_{X \in \mathcal{H}} X^{\blacktriangle} \quad \text{and} \quad \bigwedge_{X \in \mathcal{H}} X^{\blacktriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle} \Big)^{\blacktriangledown\blacktriangle}$$

(c) The maps $X^{\blacktriangle} \mapsto X^{\blacktriangle\blacktriangledown}$ and $X^{\blacktriangledown} \mapsto X^{\blacktriangledown\blacktriangle}$ are isomorphisms between $\wp(U)^{\blacktriangle}$ and $\wp(U)^{\blacktriangledown}$ — these are now also **self-dual**

Distributivity and modularity

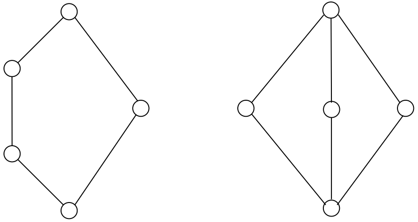
A lattice is **distributive** if for all x, y, z:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice is distributive iff none of its sublattices is isomorphic to M_3 or N_5.
A **modular lattice** is a lattice that satisfies the condition:

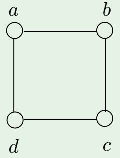
$$x \leq b \text{ implies } x \vee (a \wedge b) = (x \vee a) \wedge b.$$

A lattice L is modular iff none of its sublattices is isomorphic to N_5.

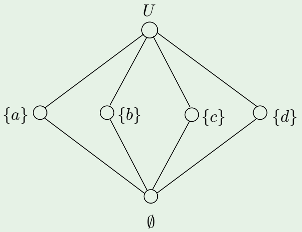


N_5 M_3

Example



Tolerance R

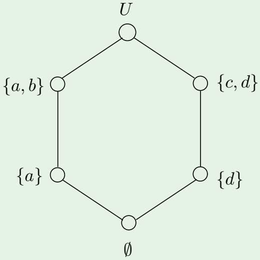


$\wp(U)^{\blacktriangledown}$

$\Longrightarrow$ The lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not always distributive

Example

Tolerance R



$\wp(U)^{\blacktriangledown}$

$\Longrightarrow$ The lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not always modular

Approximation lattices as ortholattices

An **ortholattice** is a bounded lattice equipped with an orthocomplementation:

(O1) $x \leq y$ implies $y^{\perp} \leq x^{\perp}$

(O2) $x^{\perp\perp} = x$

(O3) $x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$

Lemma

Let R be a tolerance

(a) $\wp(U)^{\blacktriangle}$ *is an ortholattice such that* $^{\perp} \colon X^{\blacktriangle} \mapsto X^{\blacktriangle c \blacktriangle}$

(b) $\wp(U)^{\blacktriangledown}$ *is an ortholattice such that* $^{\top} \colon X^{\blacktriangledown} \mapsto X^{\blacktriangledown c \blacktriangledown}$

Proposition

A complete lattice L forms an ortholattice if and only if there exists a set U and a tolerance R on U such that $L \cong \wp(U)^{\blacktriangledown} \cong \wp(U)^{\blacktriangle}$.

Irredundant coverings

A collection $\mathcal{H}$ of nonempty subsets of U is called a **covering** of U if $\bigcup \mathcal{H} = U$.

A covering $\mathcal{H}$ is **irredundant** if $\mathcal{H} \setminus \{X\}$ is not a covering for any $X \in \mathcal{H}$.

Each covering $\mathcal{H}$ of U defines a tolerance $\bigcup\{X^2 \mid X \in \mathcal{H}\}$, called the **tolerance induced** by $\mathcal{H}$.

Proposition

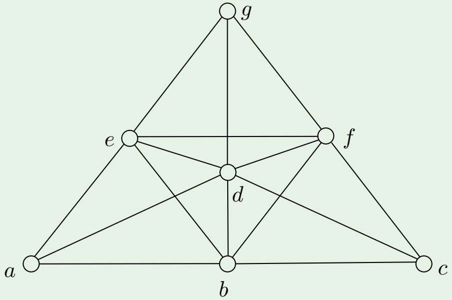
Let R be a tolerance induced by a covering $\mathcal{H} \subseteq \wp(U)$. Then, the following assertions are equivalent:

(a) *$\mathcal{H}$ is an irredundant covering;*

(b) $\mathcal{H} \subseteq \{R(x) \mid x \in U\}$

Example

Any tolerance R on U determines an undirected graph $\mathcal{G} = (U, R)$.



Tolerance R

The family $\mathcal{H} = \{\{a, b, d, e\}, \{b, c, d, f\}, \{d, e, f, g\}\}$ induces R. This covering $\mathcal{H}$ is irredundant, because $R(a) = \{a, b, d, e\}$, $R(c) = \{b, c, d, f\}$, $R(g) = \{d, e, f, g\}$.

Definition

1. A complete lattice L satisfies the **join-infinite distributive law (JID)** if for any $S \subseteq L$ and $x \in L$,

$$x \wedge \left(\bigvee S\right) = \bigvee\{x \wedge y \mid y \in S\}. \tag{JID}$$

2. The **meet-infinite distributive law (MID)** is defined:

$$x \vee \left(\bigwedge S\right) = \bigwedge\{x \vee y \mid y \in S\}. \tag{MID}$$

3. A complete lattice L is **completely distributive** if arbitrary joins distribute over arbitrary meets

Proposition

For a tolerance R on U, the isomorphic complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are completely distributive if and only if R is induced by an irredundant covering of U.

Definition

- A bounded lattice is **complemented** if every element a has a complement a': $a \vee a' = 1$ and $a \wedge a' = 0$.
- A complement is unique if the lattice is distributive
- **Boolean lattice**: distributive and complemented lattice
- **Boolean algebra**: $(B, \vee, \wedge, ', 0, 1)$

Remark

A distributive ortholattice is a Boolean lattice. Each Boolean lattice is trivially an ortholattice.

Blocks of a tolerance

- A nonempty subset X of U is an R-**preblock** if $X^2 \subseteq R$.
- An R-**block** is a maximal R-preblock.
- The relation R is completely determined by its blocks, i.e., $a\,R\,b$ if and only if there exists a block B such that $a, b \in B$.

Lemma

If R is a tolerance induced by an irredundant covering $\mathcal{H}$, then

$$\mathcal{H} = \{R(x) \mid R(x) \text{ is a block}\}.$$

For all $x \in U$, $R(x)$ is an R-block if and only if $R(x)$ is an R-preblock, i.e. a **clique** of the graph $\mathcal{G} = (U, R)$.

Let L be a lattice with a least element 0. The lattice L is **atomistic**, if any element of L is the join of atoms below it. It is well known that a complete Boolean lattice is atomistic if and only if it is completely distributive.

Proposition

Let R be a tolerance induced by an irredundant covering of U.

(a) $\wp(U)^{\blacktriangle}$ *and* $\wp(U)^{\blacktriangledown}$ *are atomistic Boolean lattices*

(b) $At(\wp(U)^{\blacktriangle}) = \{R(x) \mid R(x) \text{ is a block}\}$

(c) $At(\wp(U)^{\blacktriangledown}) = \{R(x)^{\blacktriangledown} \mid R(x) \text{ is a block}\}$

Topological spaces I

A **topological space** $(U, \mathcal{T})$ consists of a set U and a family $\mathcal{T} \subseteq \wp(U)$ such that

(TS1) $\emptyset \in \mathcal{T}$ and $U \in \mathcal{T}$,

(TS2) $X \cap Y \in \mathcal{T}$ for any sets $X, Y \in \mathcal{T}$, and

(TS3) $\bigcup \mathcal{H} \in \mathcal{T}$ for any subfamily $\mathcal{H} \subseteq \mathcal{T}$.

The family $\mathcal{T}$ is called a **topology** on U and the members of $\mathcal{T}$ are *open sets*. The complement of an open set is called a *closed set*

An operator $C\colon \wp(U) \to \wp(U)$ is a **Kuratowski closure operator** if for any $X, Y \subseteq U$,

(K1) $X \subseteq C(X)$,

(K2) $C(C(X)) = C(X)$,

(K3) $C(X \cup Y) = C(X) \cup C(Y)$, and

(K4) $C(\emptyset) = \emptyset$.

Topological spaces II

- If $\mathcal{T}$ is a topology on U, then the operator defined by

$$C(X) = \bigcap \{B \mid X \subseteq B \text{ and } B \text{ is closed}\}$$

  is a Kuratowski closure operator.
- Conversely, for a Kuratowski closure operator C on U, the family

$$\{C(X) \mid X \subseteq U\}$$

  determines a topological space whose closed sets are exactly these sets.
- **Kuratowski closure operators** are in 1-to-1 correspondence with topologies.

Heyting algebras of topologies

- A **Heyting algebra** L is a bounded lattice such that for all $a, b \in L$, there is a greatest element x of L with $a \wedge x \leq b$.
- This element is the **relative pseudocomplement** of a with respect to b, and is denoted $a \to b$.
- A complete lattice is a Heyting algebra if and only if it satisfies (JID). Then,

$$a \to b = \bigvee \{c \mid a \wedge c \leq b\}$$

- Since $\mathcal{T}$ is closed under arbitrary unions and finite intersections, the complete lattice $(\mathcal{T}, \subseteq)$ satisfies (JID): for all $\mathcal{H} \subseteq \mathcal{T}$,

$$X \cap (\bigcup \mathcal{H}) = \bigcup \{X \cap Y \mid Y \in \mathcal{H}\}.$$

Thus, every topology $\mathcal{T}$ determines a Heyting algebra

Properties of rough approximations: **quasiorders**

An **Alexandrov topology** is a topology $\mathcal{T}$ that contains also all arbitrary intersections of its members. Let $\mathcal{T}$ be an Alexandrov topology $\mathcal{T}$ on U. Then, for each $X \subseteq U$, there exists the **smallest neighbourhood**

$$N_{\mathcal{T}}(X) = \bigcap\{Y \in \mathcal{T} \mid X \subseteq Y\}.$$

In particular, the smallest neighbourhood of a point $x \in U$ is denoted by $N_{\mathcal{T}}(x)$. The family

$$\mathcal{B}_{\mathcal{T}} = \{N_{\mathcal{T}}(x) \mid x \in U\}$$

is the **smallest base** of the Alexandrov topology $\mathcal{T}$. This means that every member X of $\mathcal{T}$ can be expressed as a union of some (or none) elements of $\mathcal{B}_{\mathcal{T}}$, i.e. $X = \bigcup\{N_{\mathcal{T}}(x) \mid x \in X\}$. In addition, $\mathcal{B}_{\mathcal{T}}$ is smallest such set.

Complete lattices of Alexandrov topologies

- Every Alexandrov topology $\mathcal{T}$ defines a complete lattice:

$$\bigvee \mathcal{H} = \bigcup \mathcal{H} \quad \text{and} \quad \bigwedge \mathcal{H} = \bigcap \mathcal{H}$$

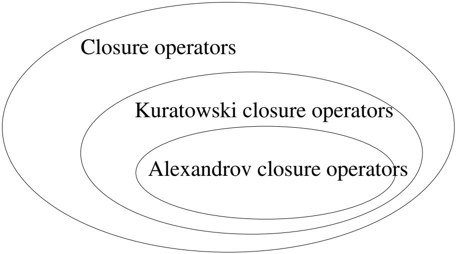
  for all $\mathcal{H} \subseteq \mathcal{T}$
- $(\mathcal{T}, \subseteq)$ is a distributive lattice
- In a complete lattice L, an element a is **completely join-irreducible** if $a = \bigvee S$ implies $a \in S$ for every $S \subseteq L$.
- The set of completely join-irreducible elements of $\mathcal{T}$ is $\mathcal{J} = \{N(x) \mid x \in U\}$.
- The lattice $\mathcal{T}$ is **spatial**, i.e. each element can be given as a join of join-irreducibles.

Alexandrov closure operator

- We say that a closure operator is an **Alexandrov closure operator** if it satisfies for all $\mathcal{H} \subseteq \mathcal{T}$,

$$C(\bigcup \mathcal{H}) = \bigcup C(\mathcal{H})$$

- As in case of topologies and Kuratowski closure operators, there is 1-to-1 correspondence between Alexandrov topologies and Alexandrov closure operators.



Alexandrov topologies and quasiorders

- There is a 1-to-1 correspondence between quasiorders and Alexandrov topologies.
- For a quasiorder R on the set U, we can define an Alexandrov topology $\mathcal{T}_R$ on U consisting of all "R-closed" subsets of U with respect to the relation R:

$$\mathcal{T}_R = \{A \subseteq U \mid (\forall x, y \in U)\, x \in A \;\;\&\;\; x\, R\, y \Longrightarrow y \in A\}$$

Alexandrov topologies and quasiorders

- The set $R(x)$ is the smallest neighbourhood of the point x in the Alexandrov topology $\mathcal{T}_R$
- Trivially, $y \in R(x)$ if and only if $x \, R \, y$.
- This hints how we may determine quasiorders by means of Alexandrov topologies
- If $\mathcal{T}$ is an Alexandrov topology, then the quasiorder $R_{\mathcal{T}}$ is defined by
$$x \, R_{\mathcal{T}} \, y \iff y \in N_{\mathcal{T}}(x).$$
- The correspondences $R \mapsto \mathcal{T}_R$ and $\mathcal{T} \mapsto R_{\mathcal{T}}$ are 1-to-1.

Alexandrov topologies and quasiorders

- For a quasiorder R, the rough approximations satisfy for all $X \subseteq U$:

$$X^{\blacktriangle\triangledown} = X^{\blacktriangle}, \; X^{\vartriangle\blacktriangledown} = X^{\vartriangle}, \; X^{\blacktriangledown\vartriangle} = X^{\blacktriangledown}, \; X^{\triangledown\blacktriangle} = X^{\triangledown}.$$

- The approximations determine two Alexandrov topologies:

$$\wp(U)^{\blacktriangle} = \wp(U)^{\triangledown} \quad \text{and} \quad \wp(U)^{\blacktriangledown} = \wp(U)^{\vartriangle}$$

- Note that $\wp(U)^{\blacktriangledown}$ is the same as $\mathcal{T}_R$ above (R-closed subsets)
- Clearly, these topologies are dual, i.e. for all $X \subseteq U$,

$$X \in \wp(U)^{\blacktriangle} \iff X^c \in \wp(U)^{\blacktriangledown}$$

Alexandrov topologies and quasiorders

For the Alexandrov topology $\wp(U)^{\blacktriangle} = \wp(U)^{\triangledown}$:

(i) ${}^{\blacktriangle} \colon \wp(U) \to \wp(U)$ is the smallest neighbourhood operator.

(ii) ${}^{\vartriangle} \colon \wp(U) \to \wp(U)$ is the Alexandrov closure operator. Note that the family of closed sets for the topology $\wp(U)^{\blacktriangle}$ is $\wp(U)^{\blacktriangledown}$; — and vice versa.

(iii) ${}^{\triangledown} \colon \wp(U) \to \wp(U)$ is the Alexandrov interior operator, i.e., it maps each set to the greatest open set contained into the set in question.

(iv) The set $\{\,\{x\}^{\blacktriangle} \mid x \in U\} = \{R^{-1}(x) \mid x \in U\}$ is the smallest base.

Alexandrov topologies and quasiorders

Similarly, for the topology $\wp(U)^{\blacktriangledown} = \wp(U)^{\triangle}$:

(i) ${}^{\triangle} \colon \wp(U) \to \wp(U)$ is the smallest neighbourhood operator.

(ii) ${}^{\blacktriangle} \colon \wp(U) \to \wp(U)$ is the Alexandrov closure operator.

(iii) ${}^{\blacktriangledown} \colon \wp(U) \to \wp(U)$ is the Alexandrov interior operator.

(iv) The set $\{\, \{x\}^{\triangle} \mid x \in U\} = \{R(x) \mid x \in U\}$ is the smallest base.

Lattice structures of approximations: **equivalences**

- Equivalence E is a tolerance **and** a quasiorder.
- For an equivalence E, $X^{\blacktriangle\blacktriangledown} = X^{\blacktriangle}$ and $X^{\blacktriangledown\blacktriangle} = X^{\blacktriangledown}$.
- Therefore, $\wp(U)^{\blacktriangledown} = \wp(U)^{\blacktriangle}$ forms a completely distributive Boolean lattice — in fact, a complete field of sets.
- The equivalence classes of E are the atoms.